Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

THE MEAN OF A FUNCTIONAL OF ARBITRARY ELEMENTS.

BY NORBERT WIENER.

1. P. J. Daniell* has recently developed a powerful method whereby if the notion of integration is once defined for a very restricted set of functions of arbitrary elements, it can be extended to a much more comprehensive set of functions. Daniell himself in his second article applied his method to the discussion of integrals in a denumerable infinity of dimensions. Daniell's method, however, leaves the mode of establishing integration over the original restricted set in general undetermined. It is the purpose of this paper to develop a method of setting up a Daniell integral which is applicable to a large group of cases, and in particular to functionals.†

2. **Definitions.** If K be any class, we shall define a *division* of K as a finite set of non-null sub-classes exhausting K at least once. Divisions of K will be represented by Greek letters with the suffix K—thus α_K, β_K, etc. A division depending on a parameter n will be represented by some such symbol as $\alpha_K(n)$. The sub-classes or intervals of a division α_K will be represented by $t_1(\alpha_K)$, $t_2(\alpha_K)$, $\cdots$, $t_m(\alpha_K)$.

A *weighted division* of K will be defined as a division of K to each term of which is assigned a positive number—its *weight*. A weighted division corresponding to α_k will be represented by such a symbol as $\alpha_K{}^U$, $\alpha_K{}^V$, or $\alpha_K{}^W$. The weight of $t_j(\alpha_K)$ will be written $U_\alpha\{t_j(\alpha_K)\}$, $V_\alpha\{t_j(\alpha_K)\}$, or $W_\alpha\{t_j(\alpha_K)\}$, respectively. $U_{\alpha(n)}$, may be abbreviated to U_n if a specific sequence of α's is understood.

A *K-partition*‡ P_K is a sequence of weighted divisions $\alpha_K{}^{U_1}(1)$, $\alpha_K{}^{U_2}(2)$, $\cdots$, $\alpha_K{}^{U_n}(n)$, $\cdots$, such that

(1) Every member of $\alpha_K(n+1)$ is wholly included in one member of $\alpha_K(n)$ and one only.

(2) $U_n\{t_j(\alpha_K(n))\}$ is the sum of all the numbers $U_{n+1}\{t_l(\alpha_K(n+1))\}$ such that $t_l(\alpha_K(n+1))$ is contained in $t_j(\alpha_K(n))$.

(3) If S_n is the 'sum' of a number of intervals from $\alpha_K(n)$, whatever n may be, and if S_{n+1} is always entirely included in S_n, then either there is a K-element common to every S_n, or the sum of the weights of the intervals in S_n approaches 0 as n grows without limit.

---

* P. J. Daniell, Annals of Mathematics, vol. 19 (1918), vol. 20 (1919).

† Cf. P. Lévy, Comptes Rendus, Aug., 1919.

‡ This notion is related to E. H. Moore's 'development.'

$\alpha_K(n)$, $\alpha_K{}^{U_n}(n)$, and $t_j(\alpha_K(n))$ are said to *belong* to P_k.

A function f defined for all elements of K is said to be a P_K *step-function* if there is an $\alpha_K(n)$ belonging to P_K such that the function is constant for every $t_j(\alpha_K(n))$. We shall say that f has degree n.

The *mean** of a P_K step-function f of degree n, taken over K with respect to P_K, is said to be

$$\frac{\sum\limits_j U_n\{t_j(\alpha_K(n))\}f(x_j)}{\sum\limits_j U_n\{t_j(\alpha_K(n))\}},$$

where x_j is a member of $t_j(\alpha_K(n))$. It will be written briefly $M_{P_K}(f)$.

A function f will be said to *have uniform P_K-continuity* if, given any positive number ϵ, there is an integer n such that if x and y belong to the same $t_j(\alpha_K(n))$, $|f(x) - f(y)| < \epsilon$.†

3. **Application of Daniell's Results.** Daniell's theory of integration is based on the existence of a set T_0 of bounded functions, which shall be closed with respect to multiplication by a constant, the addition of two functions, and the operation of taking the modulus. The class of all P_K step-functions clearly has all these properties. Daniell further postulates a finite functional operation I defined over T_0 and satisfying the conditions

$$\text{(C)}\quad I(cf) = cI(f), \text{ if } c \text{ is any constant,}$$

$$\text{(A)}\quad I(f_1 + f_2) = I(f_1) + I(f_2),$$

$$\text{(P)}\quad I(f) \geqq 0 \text{ if } f(p) \geqq 0 \text{ for all } p,$$

$$\text{(L)}\quad \text{If } f_1 \geq f_2 \geq \cdots \geq 0 = \lim f_n \text{ for every } p,$$

$$\lim I(f_n) = 0.$$

Our operation M_{P_K} satisfies all these conditions. The first three need no proof, being matters merely of elementary algebra. (L) may be proved if we can show that for every positive number a, the total weight of the set of intervals containing the K-elements p for which $f_n(p) \geq a$ approaches 0 as n grows indefinitely. Consider the set S_n of elements p for which $f_n(p) \geq a$. Clearly S_{n+1} is included in S_n. Three conceivable possibilities are open.

---

* The use of mean instead of integral is found in the posthumous papers of Gateaux (Bull. de la Soc. Math. de France, 1919). This was however unknown to me at the time I wrote this article.

† If a term x is shared by two or more members of some $\alpha(n)$, in determining functions over K, we regard x as a set of different members of K, each consisting of x *qua* member of all of the intervals of a sequence $t_1(\alpha_K(1))$, $t_2(\alpha_K(2))$,$\cdot \cdots$, where $t_{n+1}(\alpha_K(n+1))$ is contained in $t_n(\alpha_K(n))$. Each $t_n(\alpha_K(x))$ will contain only this value of x. It may be shown (in the general case, with the aid of Zermelo's axiom) that this change will not affect the validity of the three conditions for a division, and will render every step-function single-valued.

(1) Every S_n contains intervals from some fixed $\alpha_K(m)$. In this case there is some term p common to every S_n, so that for every $n, f_n(p) \geqq a$. This however is contrary to our hypothesis.

(2) (1) is not satisfied, but there is some p common to every S_n. This again violates our hypothesis.

(3) Neither (1) nor (2) is satisfied. Then by (3) of the definition of a partition, the sum of the weights of the intervals in S_n approaches 0 as n grows without limit. In this case it follows from the definition of M_{P_K} that $\lim_{n=\infty} M_{P_K}(f_n) \leqq a$.

Hence (L) is proved, and we are at once in a position to apply Daniell's results. He defines T_1 as the class of all functions f which are the limits of a sequence f_n where every f_n belongs to T_0 and $f_1 \leqq f_2 \leqq \cdots$. $I(f)$ is defined as the limit of $I(f_n)$. Given any function f, $\dot{I}(f)$ is defined as the lower bound of $I(\varphi)$ for all functions φ of class T_1 such that $\varphi \geqq f$, and $I(f)$ as $-\dot{I}(-f)$. If $\dot{I}(f) = I(f)$ and is finite, f is said to be summable, and it is proved that if $f_1, f_2, \cdots, f_n, \cdots$, is a sequence of summable functions with limit f, and if a summable function φ exists such that $|f_n| \leqq \varphi$ for all n, f is summable, and $\lim I(f_n) = I(f)$. We can translate all this into our language, and in particular we can say that if a function can be obtained as a limit of a set of P_K step-functions all less in modulus than a certain P_K step-function, it is summable and its mean may be determined. A constant is clearly a P_K step-function, and we neither gain nor lose any generality by insisting that all our P_K step-functions be less in modulus than some constant.

Every bounded uniformly P_K-continuous function is summable. For let $f(x)$ be such a function and let $f_n(x)$ be the maximum of $f(x)$ in an interval $t_j(\alpha_K(n))$ that contains x and belongs to P_K. It is clear that by definition f_n is a P_K step-function, and furthermore that $|f_n(x)| < 1 + \max |f(x)|$. Moreover, since f is uniformly P_K-continuous, and since $f_n(x) = f(x_n)$ where x and x_n lie in the same $t_j(\alpha_k(n))$, $\lim_{n=\infty} |f(x) - f_n(x)| = 0$, so that $f(x) = \lim f_n(x)$.

An important generalization of this theorem is the following: *if L_m is the set of all K-elements contained in a finite number of intervals of P_K, if L_m is contained in L_{m+1} for all m and if every K-element is in some L_m, then if f is bounded and uniformly continuous over all the intervals of each L_m, it is summable according to M_{P_K}.* Let $g_m(x) = f(x)$ when x is in L_m and 0 otherwise. Every g_m is summable by the argument of the last paragraph; and the whole set is bounded as no g_m can be larger than f in its largest modulus. As f is the limit of $\{g_m\}$, by Daniell's theorem, f is summable.

4. **Examples.** (*a*) K is the set of all real numbers in the closed interval (a, b). $\alpha_K(n)$ is the set of intervals

$$\left\{\frac{a + h(b - a)}{2^n}, \qquad \frac{a + (h + 1)(b - a)}{2^n}\right\} \qquad h < 2^n$$

and each interval of $\alpha_K(n)$ is given the weight $1/2^n$. The mean $M_{P_K}(f)$ is the Lebesgue integral $1/(b - a)\int_a^b f(x)dx$, which is what we should naturally call the mean of $f(x)$ over the interval (a, b). This results immediately from the facts (1) that any continuous function can be obtained as the limit of a bounded sequence of step-functions constant over all the intervals of some $\alpha_K(n)$; (2) that, as Daniell has shown, every Lebesgue summable function is summable in his sense if $I(f)$ be Riemann integration and T be the set of all continuous functions, and hence if $I(f)$ be the Riemann integral confined to some set of functions with respect to which all continuous functions are summable.

(*b*) K is the set of all points $(x_1, x_2, \cdots, x_n)$ in a bounded region V of n-space of 'volume' v. $\alpha_K(l)$ is the set of all intervals

$$a_k/2^l \leq x_k \leq (a_k + 1)/2^l \qquad (1 \leq k \leq n),$$

where the a_k's are integers ranging between bounds not less than the largest coördinate of any point in V. The weight of each interval of $\alpha_K(l)$ is the limit of the sum of the 'volumes' of the intervals of $\alpha_K(m)$ contained in this interval to the sum of the 'volumes' of all intervals of $\alpha_K(m)$, as m grows indefinitely, and V shall be such that this limit always exists. $M_{P_K}(f)$ becomes

$$(1/v)\int\int_V \cdots \int f(x_1, x_2, \cdots, x_n)dV.$$

(*c*) K is the set of all points $(x_1, x_2, \cdots, x_m, \cdots)$ in a region of space of a denumerably infinite number of dimensions such that $a_m \leq x_m \leq b_m$ for all m. $\alpha_K(n)$ is the set of regions

$$\frac{2^{n-m}a_m + h_m(b_m - a_m)}{2^{n-m}} \leq x_m \leq \frac{2^{n-m}a_m + (h_m + 1)(b_m - a_m)}{2^{n-m}}$$

for all $m \leq n$. (Here h_m is some integer between 0 and $2^{n-m} - 1$). The weight of each region is

$$\frac{1}{2^{n(n+1)/2}}.$$

In the case where for every m $a_m = 0$ and $b_m = 1$, $M_{P_K}(f)$ is Daniell's

$$I(f) = \int_0^1 \cdots \int_0^1 \cdots f(p)dx_1dx_2 \cdots dx_n \cdots,$$

and is defined for a class of functions including all continuous functions. This results from the fact that every function belonging to Daniell's T_0, the class of all continuous functions of a finite number of variables, is summable in our system.

(*d*) K is the set of all continuous functions defined between 0 and 1, satisfying a Lipschitz condition and themselves lying between the bounded continuous functions $\varphi(x)$ and $\psi(x)$. The coefficient of irregularity of a function f (written $c(f)$) is the upper bound of the modulus of the slope of a chord of the curve representing the function. $\alpha_K(n)$ is the set of regions each of which consists of all continuous functions f satisfying simultaneously the 2^n pairs of inequalities

$$\frac{2^{2^n-m}\varphi\left(\frac{m}{2^n}\right) + h_m\left\{\psi\left(\frac{m}{2^n}\right) - \varphi\left(\frac{m}{2^n}\right)\right\}}{2^{2^n-m}} \leq f\left(\frac{m}{2^n}\right)$$

$$\leq \frac{2^{2^n-m}\varphi\left(\frac{m}{2^n}\right) + (h_m + 1)\left\{\psi\left(\frac{m}{2^n}\right) - \varphi\left(\frac{m}{2^n}\right)\right\}}{2^{2^n-m}}$$

and an inequality either of the form

$$k_m \leq c(f) \leq k_m + 1 \qquad \text{or} \qquad c(f) \geq m$$

for all integral values of m not greater than 2^n. Here h_m is an integer between 0 and 2^{2^n-m}, and k_m is an integer between 0 and $m - 1$. The coefficient of irregularity of an interval is defined as the smallest integer not less than 2 that is greater than the coefficient of irregularity of some function in the interval. The weighting of $\alpha_K(n)$ is carried on in a progressive manner as follows: when K is divided into intervals of $\alpha_K(1)$ or when any interval of $\alpha_K(n)$ is divided into intervals of $\alpha_K(n + 1)$, all intervals with the same coefficient of irregularity are weighted alike. The total weight of all the sub-intervals with a given coefficient of irregularity c greater than that of the original interval we shall then make $w/c!$, where w is the weight of the original interval. The rest of the weight, of course, goes to those sub-intervals with the same coefficient of irregularity as the original interval. If w is the original weight of K, and q the least coefficient of irregularity of any function it contains, the total weight of those intervals in any division whose coefficients of irregularity exceed k may be shown to be no greater than $w(e - 2)^k$, if $k > q$.

That is, the total weight of any set of intervals whose coefficient of irregularity is greater than a given number N approaches 0 as N grows without limit. Furthermore, if $S_1, S_2, \cdots S_n, \cdots$ are sets of intervals such that every S_n contains intervals with a coefficient of irregularity not greater than a fixed number N, and if S_{n+1} is always contained in S_n, there is always a continuous function belonging to every S_n. To prove this, in the first place, if we discard from S_n all intervals whose coefficient of irregularity equals or exceeds N, and call the remaining set R_n, R_{n+1} will be contained in R_n, which will always exist, whatever n. Let us divide the intervals of R_1 into those that contain intervals from an infinity of R_n's and those that do not. Clearly there will be at least one interval in the former class. Select one such interval.* This will contain intervals belonging to some R_k. These intervals can again be divided into two classes, and we can again select those that contain sub-intervals of an infinite sequence of orders. In this way an infinite chain $t_1, t_2, \cdots, t_n, \cdots$ can be selected of intervals belonging respectively to $S_1, S_k, \cdots, S_l, \cdots$, and all of a coefficient of irregularity no greater than N. From a certain stage on these t's will contain no function whose coefficient of irregularity is more than $N + 1$. By a theorem of Frechet,† since t_1 is a bounded class of equally continuous functions, it will be compact. It may readily be proved that every t_n is closed and hence extremal.‡ Consequently there will be a continuous function f common to every t_n, and therefore to every S_n. This completes the proof of (3) of the definition of a partition. The satisfaction of (1) and (2) is immediately obvious. We are hence in a position to apply our definition of a mean to functionals of continuous functions, and to give a meaning of $M_{P_{\bar{K}}}\{F\}$.

It should be noted that there is much that is arbitrary in the actual carrying out of this definition. What is really essential is that some coefficient of irregularity be chosen so that every "Einschachtelung" of intervals of less than a given coefficient of irregularity should contain a continuous function, and that then a method of weighting be adopted which shall make the total weight of the set of intervals whose coefficient of irregularity is greater than N a decreasing function of N that approaches 0 as a limit. This method can at once be extended to space-curves, surfaces, and all such entities as are usually made the arguments of functionals.

It clearly follows by the theorem at the end of § 3 that if a functional is bounded and is uniformly P_L-continuous over every P_L that consists of all the $\alpha_K(n)$'s restricted to functions of no more than a given coefficient of irregularity, it is summable with respect to M_{P_K}. Now, the set of all functions lying between two given functions in modulus and of no more than a given coefficient of irregularity is extremal, as may be proved from the fact that it is bounded and equally continuous. Hence by a theorem of Fréchet, every bounded functional F continuous over the set is uniformly continuous over the set in Fréchet's sense. This means that for every δ there is an ϵ independent of f and g such that if

---

* An ordinal arrangement of every $\alpha_K(n)$ can be found which will make this and the following selections perfectly determinate, and will consequently avoid the difficulties of Zermelo's axiom.

† M. Fréchet, Rendiconti del Circolo Matematico di Palermo, vol. 22 (1906), p. 37.

‡ Ibid., p. 7.

$$|f(x) - g(x)| < \epsilon$$

for every x,

$$|F(f) - F(g)| < \delta$$

Every functional uniformly continuous in his sense is uniformly continuous in ours, for if two functions f and g have coefficients of irregularity less than or equal to N and if

$$|f(x) - g(x)| < \epsilon$$

for all the points between $x = 0$ and $x = 1$ for which $x = a/2^n$, then

$$|f(x) - g(x)| < \epsilon + \frac{Na}{2^{n-1}}$$

for all points. Since $\epsilon + Na/2^{n-1}$ becomes smaller and smaller as we constrain f and g to lie within smaller and smaller intervals of P_K, the dependence of our definition of uniform continuity on that of Fréchet follows. Hence every bounded continuous functional is summable in accordance with our definition.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY,
November 5, 1919.